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THE INITIAL VALUE FOR NON-LINEAR BOUNDARY VALUE PROBLEM

A. G. Patil
Department of Mathematics, Kisan Arts Commerce and Science College, Parola, Dist Jalgaon.(M.S.)

ABSTRACT:

The aim of this work is to demonstrate that within a similarity approach some non-linear boundary value problem (BVP) governed by ordinary differential equations can be transformed into initial value problem (IVP). The Infinitesimal Lie group method applied to derive proper transformations that transforms BVP into IVP. The non-linear initial value problem is solved by 4^{th} order Runge-Kutta method. The numerical results are in agreement with those available in literature.

Key Words :

Initial value method, Lie group method, infinitesimal transformation, 4^{th} order Runge-Kutta Method.

INTRODUCTION :

The boundary layer theory for non-Newtonian power law fluids was probably first develop and discussed by Schowalter [1]. Later on numerous contribution in this field was made by may other research scholars like Skelland [2], Na et al , [3] Timol et al [4] etc., Hsu [5] has derived series expansion for equation of motion governing the boundary layer flow of non-Newtonian power law fluids past a semi-infinite series. Non-linear ordinary differential equations of boundary value type occur rather frequently in various fields of science and engineering. There are various methods for solving this type of equations.

The method of transformation from boundary value problems to initial value problems was first introduced by Toepfer [6] then extended by Klamkin [7] and Na [8]. But the group-theoretic concepts in such transformations were first introduced by Na [9]. These group methods are more versatile and gives more realistic result compared to the other techniques as these methods are based on the simple group invariance properties of the boundary value problems.

There are several group theoretic techniques like Inspectional group method, Infinitesimal group method, Spiral group theoretic method etc. are available for such transformations. By these methods the boundary value problems can be transformed into initial value problem which can be easily solved by numerical procedure.

In the present paper the method of solution of non-linear boundary value problem governing the motion of steady two-dimensional incompressible boundary layer flow of non-Newtonian power-low fluids is discussed. This type of equation is known as generalized Blasius equation which is highly non-linear boundary value problem. Further the presence of Power-Law fluids model make problem much complex. Using similarity transformations these equations governing the flows are first transformed into ordinary differential equation along with the related boundary conditions. The invariance of such boundary value problem under the infinitesimal group method leads to a particular form of the characteristic function w [10]. After finding the value of w, the missing boundary condition is obtained. In the inspectional group method, the differential equation can be independent of the parameter of transformation and the parameter of transformation is identified as the 'missing' boundary condition. Using this condition the initial value problem is solved by 4th order Runge-Kutta Method.

PROBLEM FORMULATION :

For a steady, two-dimensional, incompressible, laminar flow past a semi-infinite flat plate, the governing boundary layer equation of continuity and motion are

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\tau_{xy}}{\partial y} \tag{2}$$

Also, the shear stress equation is

$$\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n \tag{3}$$

The boundary conditions to be satisfied in this problem are:

$$\text{at} \quad y = 0: \quad u = 0, \quad v = 0 \tag{4}$$

$$\text{and} \quad y \to \infty: \quad u \to U = cons\tan t \tag{5}$$

Introducing following similarity transformations in equations (1)-(5),

$$\eta = yU\left[\frac{K}{\rho}n(n+1)U^{2n-1}x\right]^{-\frac{1}{n+1}} \Psi(x,y) = \left[\frac{K}{\rho}n(n+1)U^{2n-1}x\right]^{\frac{1}{n+1}} f(\eta) \tag{6}$$

$$\text{Where } u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x} \tag{7}$$

The continuity equation (1) gets satisfied identically and the boundary layer equation of motion (2), with the expression for shear stress (3) takes the form

$$f''' + f\left(f''\right)^{2-n} = 0 \tag{8}$$

The boundary conditions, equation (4) and (5), now becomes,

$$f(0) = 0, \quad f'(0) = 0 \tag{9}$$

$$f'(\infty) = 1 \tag{10}$$

By use of the transformations, Equation (6), the shear stress define in equation (3) can be written as

$$\tau_{xy} = \rho U^2 \left[n(n+1)\right]^{-\frac{n}{n+1}} f^n{}_{\eta\eta} N_{\mathrm{Re}\,x}^{-\frac{1}{n+1}} \tag{11}$$

Where $N_{\mathrm{Re}\,x} = \left(\rho U^{2-n} x^n\right)/K$ is the Reynolds number based on x. The co-efficient of skin friction is then defined as

$$C_d = \left[n(n+1)\right]^{-\frac{n}{n+1}} f^n{}_{\eta\eta}(0) \tag{12}$$

It is noted that the definition for C_d is exactly the same as $C(n)$ of Acrivos, et al. [11].

<u>SOLUTION</u> :

The objective is to transform the Boundary Value Problem given by equation (8) to (10) into the following Initial Value Problem:

$$\frac{d^3 g}{d\xi^3} + g\left(\frac{d^2 g}{d\xi^2}\right)^{2-n} = 0 \tag{13}$$

$$\xi = 0 : g(0) = 0, \; g'(0) = 0, \; g''(0) = 1 \tag{14}$$

We have used Infinitesimal Group Method to derive equations (13) and (14):

We will now derive the transformation using Infinitesimal Group Method.

<u>INFINITESIMAL GROUP METHOD</u> :

Let us define an infinitesimal group of transformations:

$$\begin{aligned}
\bar{\eta} &= \eta + \sum \phi(n, f) + o(\varepsilon^2) \\
\bar{f} &= f + \sum \theta(n, f) + o(\varepsilon^2) \\
\bar{p} &= p + \sum \xi(\eta, f, p) + o(\varepsilon^2) \\
\bar{q} &= q + \sum \delta(\eta, f, p) + o(\varepsilon^2) \\
\bar{r} &= r + \sum \rho(\eta, f, p) + o(\varepsilon^2)
\end{aligned} \tag{15}$$

Where,

$$p = \frac{df}{d\eta}, \quad q = \frac{d^2 f}{d\eta^2}, \quad r = \frac{d^3 f}{d\eta^3}$$

In order to express ø, θ, ξ, δ and ρ in terms of characteristic function, ω, we can write,

$$w = \phi p + \theta q + \xi r - \delta \tag{16}$$

$$\phi = \frac{\partial w}{\partial p}, \quad \theta = p\frac{\partial w}{\partial p} - w, \quad \xi = -X(w)$$

$$\delta = -\left[X^2 w + 2qX \frac{\partial w}{\partial p} + q\frac{\partial w}{\partial f} \right] \tag{17}$$

$$\rho = -\left[X^3 w + 3qX^2 \frac{\partial w}{\partial p} + 3qX\frac{\partial w}{\partial f} + 3q^2 \frac{\partial^2 w}{\partial f\, \partial p} \right] - r\left[3X\frac{\partial w}{\partial p} + \frac{\partial w}{\partial f} \right]$$

(18)

Where, $X(w) = \frac{\partial w}{\partial \eta} + p\frac{\partial w}{\partial f}$

Equation (8) can be written as ,

$$r + f q^{2-n} = 0 \tag{19}$$

Invariance of equation (19) can be written as,

$$\rho + q^{2-n}\theta + f(2-n)q^{1-n}\delta = 0 \tag{20}$$

Using equation (16 (17) and (18) in equation (20), we get,

$$-\left[X^3 w+3qX^2\frac{\partial w}{\partial p}+3qX\frac{\partial w}{\partial f}+3q^2\frac{\partial^2 w}{\partial f\,\partial p}\right]-r\left[3X\frac{\partial w}{\partial p}+\frac{\partial w}{\partial f}\right]$$
$$+q^{2-n}\left[p\frac{\partial w}{\partial p}-w\right]-f\left(2-n\right)q^{1-n}\left[X^2 w+2qX\frac{\partial w}{\partial p}+q\frac{\partial w}{\partial f}\right]=0 \qquad (21)$$

Simplifying and re-writing equation (21), we have,

$$X^3 w+\left[-3qX+3fq^{2-n}-2f\left(2-n\right)q^{2-n}\right]X\frac{\partial w}{\partial p}$$
$$+3qX\frac{\partial w}{\partial f}+3q^2\frac{\partial^2 w}{\partial f\,\partial p}-\left(n-1\right)fq^{2-n}\frac{\partial w}{\partial f}$$
$$+f\left(2-n\right)q^{1-n}X^2 w-pq^{2-n}\frac{\partial w}{\partial p}-wq^{2-n}=0 \qquad (22)$$

Since w is linear in p,

$$w=w_1\left(\eta,f\right)p+w_2\left(\eta,f\right) \qquad (23)$$

Substituting equation (23) into equation (22) and equating the co-efficient to zero we get,

$$\frac{\partial^3 w_1}{\partial\eta^3}+\frac{\partial^3 w_2}{\partial f\,\partial\eta}+2\frac{\partial^3 w_2}{\partial\eta^2\,\partial f}=0 \qquad (24)$$

$$3\frac{\partial^3 w_1}{\partial\eta^2\,\partial f}+3\frac{\partial^3 w_2}{\partial\eta\,\partial f^2}=0 \qquad (25)$$

$$3\frac{\partial^3 w_1}{\partial\eta\,\partial f^2}+\frac{\partial^3 w_2}{\partial f^3}=0 \qquad (26)$$

$$3f\frac{\partial w_1}{\partial\eta}-2f\left(2-n\right)\frac{\partial w_1}{\partial\eta}-\left(n-1\right)f\frac{\partial w_2}{\partial f}-w_2=0 \qquad (27)$$

The characteristic function can be obtained by solving above equations,

$$w = \left(c_3\eta + c_4\right)p + \eta f c_3 \tag{28}$$

Where

$$w_1 = c_3\eta + c_4 \tag{29}$$

$$w_2 = \eta f c_3 \tag{30}$$

The infinitesimals ø, θ, ξ and δ are,

$$\begin{aligned} \phi &= c_3\eta + c_4, \\ \theta &= -nfc_3 \\ \xi &= -(\eta+1)pc_3 \\ \delta &= -3qc_3 \end{aligned} \tag{31}$$

We now solve the system of equations,

$$\frac{d\eta}{\phi} = \frac{df}{\theta} = \frac{dp}{\xi} = \frac{dq}{\delta} = da \tag{32}$$

The results are,

$$\frac{q^*}{q} = e^{-3c_3 a} \tag{33}$$

$$\frac{p^*}{p} = e^{-(n+1)c_3 a} \tag{34}$$

Evaluating equation (34) at η = ξ = ∞,

$$\frac{dg}{d\xi}(\infty) = e^{-(n+1)c_3 a} \tag{35}$$

Evaluating equation (33) at η = ξ = 0,

$$\frac{d^2 f}{d\eta^2}(0) = e^{-3c_3 a} \tag{36}$$

Eliminating c_3a from equation (35) and equation (36), we obtain the relationship,

$$\frac{d^2 f}{d\eta^2}(0) = \left[\frac{dg}{d\xi}(\infty)\right]^{-\frac{3}{n+1}} \tag{37}$$

This relation is same to that derived by inspectional group method. Now, for comparison purpose the initial value problem (8)-(10) solved for the power-law index n=1 by 4th order Runge-Kutta Method. After this numerical procedure we obtain the value of $\frac{dg}{d\xi}(\infty) = 2.0625$ and then using the transformations $G : \eta = A^{-\frac{1}{3}}\xi$, $f = A^{\frac{1}{3}} g$ we get the value of group parameter A from the relationship (37), which is given by $A = \frac{d^2 f}{d\eta^2}(0) = 0.3376$ which is well agreed with that of derived by Na [9]. The variation of f vs. η and ξ vs. g are given in figure (1) and (2).

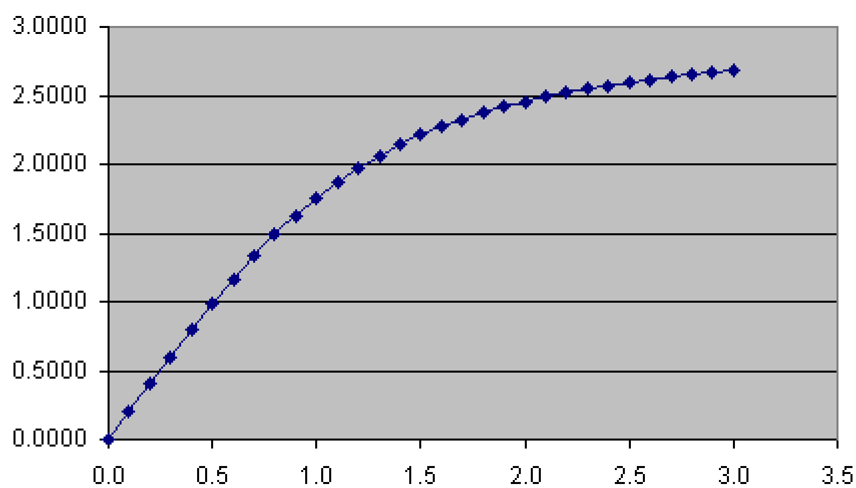

Figure 1

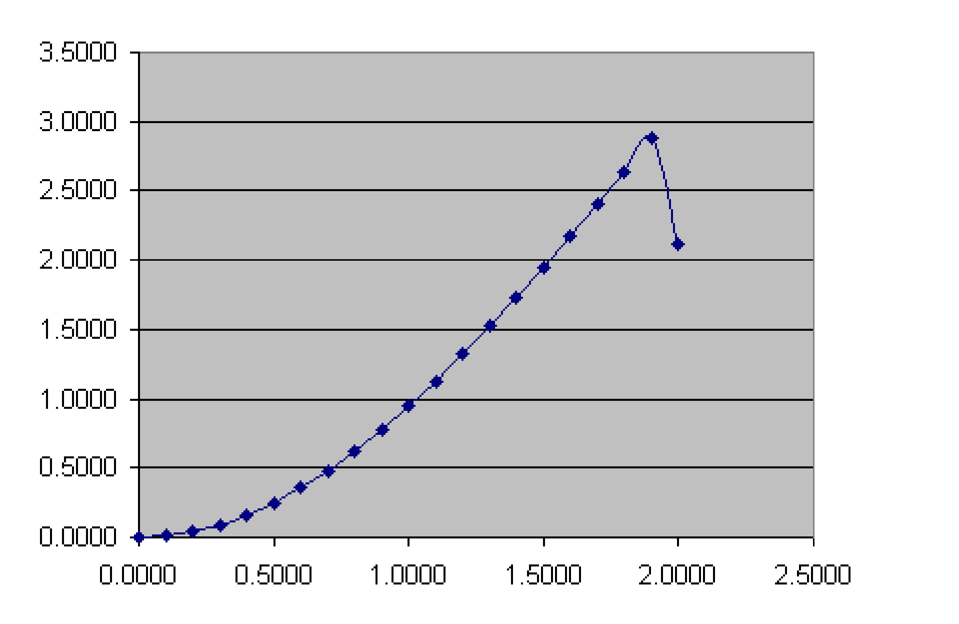

Figure 2

<u>Conclusion</u> :

Group theoretic techniques for transforming a boundary value problem, to an initial value problem were covered in this paper. Two methods for obtaining such transformations were discussed (1) The inspectional group method (2) The infinitesimal group method. The required transformation that convert a boundary value problem to an initial value problem were obtained by stipulating that (a) the governing differential equations be invariant under the group of transformations (b) the parameter of transformation is obtained as the 'missing' boundary conditions. The methods are simple and its application is straight foreword.

The basic difference between the inspectional group method and infinitesimal group method is that in the later, the transformation is deduced systematically by starting out with a general group of transformations and thus it has wide range of applications.

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